Utility Option Pricing Model (UOPM)
Two-State Model - Option Pricing in Complete Markets

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In this white paper we will use the two-state asset pricing model from Part I to price a put and call option assuming that markets are complete. Some attributes of a complete market are...

1 Every asset in every possible state of the world has a price.
2 Market where buyers and sellers trade all possible goods, and all possible contingencies are covered.
3 Possible to instantaneously enter into any position regarding any future state of the world.
4 Negligible transaction costs and perfect information.
5 All idiosyncratic risks to individuals can be insured.
6 Arbitrages are not possible.

We will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We have two possible states of the world (SOTW) at the end of the option term: SOTW = Up and SOTW = Down. We are given the following model assumptions... [1]

Table 1: Model Assumptions From Part I

Symbol	Definition	Value
S_0	Share price at time zero (\$)	20.00
B_0	Risk-free bond price at time zero (\$)	100.00
X_T	Option exercise price at time T (\$)	22.50
α	Bond continuous-time risk-free rate (%)	4.15
μ	Share continous-time return drift (%)	7.50
σ	Share continous-time return volatility (%)	18.00
p	Probability that share price will increase (%)	50.00
$(1-p)$	Probability that share price will decrease (%)	50.00
T	Option term in years (#)	3.00

Table 2: End-Of-Term Random Asset Prices From Part I

Symbol	Description	Value
$S(U)$	Share price given SOTW = Up	34.21
$S(D)$	Share price given SOTW = Down	18.34
$B(U)$	Bond price given SOTW = Up	113.26
$B(D)$	Bond price given SOTW = Down	113.26
$C(U)$	Call option price given SOTW = Up	11.71
$C(D)$	Call option price given SOTW = Down	0.00
$P(U)$	Put option price given SOTW = Up	0.00
$P(D)$	Put option price given SOTW = Down	4.16

Note: The Black-Scholes option pricing model (BSOPM) assumes a complete market. Assumptions include no transaction costs, perfect liquidity, no restrictions to short selling, and no arbitrages are possible.

Questions:

Question 1: What is the no-arbitrage call option price at time zero?
Question 2: What is the no-arbitrage put option price at time zero?
Question 3: Prove that put-call parity holds in a complete market.
Question 4: How would the Black-Scholes OPM price these options?

Option Pricing in Complete Markets

We will define the variable X_T to be the option exercise price at time T. This statement in equation form is...

$$X_T = \text{Option exercise price} \tag{1}$$

We will define the variables $C(U)$ and $C(D)$ to be call option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0, T]$. The equations for call option payoffs are...

$$C(U) = \text{Max}\left[S(U) - X_T, 0\right] \text{ ...and... } C(D) = \text{Max}\left[S(D) - X_T, 0\right] \tag{2}$$

We will define the variables $P(U)$ and $P(D)$ to be put option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0, T]$. The equations for put option payoffs are...

$$P(U) = \text{Max}\left[X_T - S(U), 0\right] \text{ ...and... } P(D) = \text{Max}\left[X_T - S(D), 0\right] \tag{3}$$

Imagine that we are short one option and want to hedge that exposure. The cost to hedge at time zero would be the price of the option at time zero. The hedge portfolio will consist of long/(short) positions in the underlying stock and a risk-free bond. We will make the following variable definitions...

Variable	Definition
$O(U)$	Option payoff at time T given that the share price random variable $z = +1$ at time T
$O(D)$	Option payoff at time T given that the share price random variable $z = -1$ at time T
$\theta(S)$	Number of shares of the underlying asset purchased/(sold) at time zero
$\theta(B)$	Number of risk-free bonds purchased/(sold) at time zero

The hedge portfolio asset weights are such that the following two equations hold at time T...

$$\theta(S)\, S_U + \theta(B)\, B_U = O(U) \text{ ...and... } \theta(S)\, S_D + \theta(B)\, B_D = O(D) \tag{4}$$

We will define matrix $\mathbf{A}$ to be a matrix of hedge portfolio asset prices at time T, vector $\mathbf{v}$ to be a vector of hedged option payoffs at time T, and vector $\mathbf{w}$ to be a vector of hedge portfolio asset weights that we must solve for. Using Equation (4) above, the equations for our matrix and vectors are...

$$\mathbf{A} = \begin{bmatrix} S(U) & B(U) \\ S(D) & B(D) \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} O(U) \\ O(D) \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}} = \begin{bmatrix} \theta(S) \\ \theta(B) \end{bmatrix} \tag{5}$$

Using Equation (5) above, we can write our system of linear equations as a matrix:vector product. The system of linear equations and the solution to hedge portfolio asset weights are...

$$\text{if... } \mathbf{A}\vec{\mathbf{w}} = \vec{\mathbf{v}} \text{ ...then... } \vec{\mathbf{w}} = \mathbf{A}^{-1}\,\vec{\mathbf{v}} \tag{6}$$

Using the equations above, the equation for no-arbitrage option price at time zero is...

$$\text{No-arbitrage option price at time zero} = \theta(S)\, S_0 + \theta(B)\, B_0 \tag{7}$$

We will define the variables C and P to be the price of a call option and put option, respectively, at time zero. In complete markets where arbitrage is not allowed, the following equation for put-call parity will hold...

$$\text{Put-call parity} = S_0 - C + P - X_T\, \text{Exp}\left\{ -\alpha\, T \right\} = 0 \tag{8}$$

Answers To Our Hypothetical Problem

Question 1: What is the no-arbitrage call option price at time zero?

Using Equation(5) above and Tables 1 and 2 above, our matrix and vector definitions are...

$$\mathbf{A} = \begin{bmatrix} 34.21 & 113.26 \\ 18.34 & 113.26 \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} 11.71 \\ 0.00 \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}} = \begin{bmatrix} \theta(S) \\ \theta(B) \end{bmatrix} \tag{9}$$

Using Equation (6) above, the solution to the hedge portfolio asset weight vector is...

$$\vec{\mathbf{w}} = \mathbf{A}^{-1}\,\vec{\mathbf{v}} = \begin{bmatrix} 0.7378 \\ -0.1195 \end{bmatrix} \tag{10}$$

Using Equations (7) and (10) above, the answer to the question is...

$$\text{No-arbitrage call option price at time zero} = 0.7378 \times 20.00 - 0.1195 \times 100.00 = 2.81 \tag{11}$$

Question 2: What is the no-arbitrage put option price at time zero?

Using Equation(5) above and Tables 1 and 2 above, our matrix and vector definitions are...

$$\mathbf{A} = \begin{bmatrix} 34.21 & 113.26 \\ 18.34 & 113.26 \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} 0.00 \\ 4.16 \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}} = \begin{bmatrix} \theta(S) \\ \theta(B) \end{bmatrix} \tag{12}$$

Using Equation (6) above, the solution to the hedge portfolio asset weight vector is...

$$\vec{\mathbf{w}} = \mathbf{A}^{-1}\,\vec{\mathbf{v}} = \begin{bmatrix} -0.2622 \\ 0.0792 \end{bmatrix} \tag{13}$$

Using Equations (7) and (13) above, the answer to the question is...

$$\text{No-arbitrage put option price at time zero} = -0.2622 \times 20.00 + 0.0792 \times 100.00 = 2.68 \tag{14}$$

Question 3: Prove that put-call parity holds in a complete market.

Using Equation(8), (11), and (14) above and Tables 1 and 2 above, the answer to the question is...

$$\text{Put-call parity} = 20.00 - 2.81 + 2.68 - 22.50 \times \text{Exp}\left\{ -0.0415 \times 3.00 \right\} = 0 \tag{15}$$

Question 4: How would the Black-Scholes OPM price these options?

Using the data above, the input parameters to the Black-Scholes Option Pricing Model are...

Description	Value
Share price	20.00
Exercise price	22.50
Risk-free rate	0.0415
Dividend yield	0.0000
Volatility	0.1800
Term in years	3.00

The option values via our two-state model (UOPM) and the Black-Scholes model (BSOPM) are...

Description	UOPM	BSOPM
Call option	2.81	2.54
Put option	2.68	2.40

References

[1] Gary Schurman, *Utility Option Pricing Model (UOPM) - The Two-State Asset Model,* December, 2023.